HW #1: 5.1: 2, 6, 12, 18, 23 aeg
A 3x3 matrix M has e.vals 1, 2, & 3. What is the coeff. of t^2 in the char. poly. of M?
5.2: 2, 6, 13 dfi   Prove Thm 5.1 1 & 2

5.1) ② $\begin{pmatrix} 7 & 5 \\ -10 & -8 \end{pmatrix} = A$

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{vmatrix} = (7-\lambda)(-8-\lambda) + 50$$
$$= -56 + \lambda + \lambda^2 + 50 = \lambda^2 + \lambda - 6$$
$$= (\lambda+3)(\lambda-2) = 0$$

$\Rightarrow$ E.vals: $\lambda_1 = -3$, $\lambda_2 = 2$

Char. Poly: $\lambda^2 + \lambda - 6 = 0$

$\underline{\lambda_1 = -3}$

$$\text{null}\begin{pmatrix} 7-(-3) & 5 \\ -10 & -8-(-3) \end{pmatrix} = \text{null}\begin{pmatrix} 10 & 5 \\ -10 & -5 \end{pmatrix} \underset{R_1/5}{\overset{R_2+R_1}{=}} \text{null}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$= \begin{pmatrix} -c \\ 2c \end{pmatrix}$, $c \in \mathbb{R}$ Let $x_2 = 2c$

So $v_1 = c\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ for any $c \in \mathbb{R}$.

$\underline{\lambda_2 = 2}$

$$\begin{pmatrix} 7-2 & 5 \\ -10 & -8-2 \end{pmatrix} \sim \begin{pmatrix} 5 & 5 \\ -10 & -10 \end{pmatrix} \underset{R_1/5}{\overset{R_2+2R_1}{\sim}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow v_2 = \begin{pmatrix} d \\ -d \end{pmatrix} = d\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for any $d \in \mathbb{R}$.

Char. Poly: $\lambda^2 + \lambda - 6 = 0$		
Evals:	$\lambda_1 = -3$	$\lambda_2 = 2$
E.vecs:	$v_1 = c\begin{pmatrix} -1 \\ 2 \end{pmatrix}$	$v_2 = d\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(6) $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = 2-3\lambda+\lambda^2+2 = \lambda^2-3\lambda+4=0$

$\Rightarrow \lambda = \dfrac{3 \pm \sqrt{9-16}}{2} = \dfrac{3 \pm \sqrt{-7}}{2}$

no real e.vals

(12) $\begin{vmatrix} -4-\lambda & 0 & 0 \\ -7 & 2-\lambda & -1 \\ 7 & 0 & 3-\lambda \end{vmatrix} = (-4-\lambda)[(2-\lambda)(3-\lambda)-0]$

$= (-4-\lambda)(6-5\lambda+\lambda^2) = -24+20\lambda-4\lambda^2-6\lambda+5\lambda^2-\lambda^3$

$= -\lambda^3+\lambda^2+14\lambda-24 = (\lambda+4)(\lambda-2)(\lambda-3)=0$

E.vals: $\lambda_1=-4,\ \lambda_2=2,\ \lambda_3=3$

$\underline{\lambda_1=-4}$

$\begin{pmatrix} 0 & 0 & 0 \\ -7 & 6 & -1 \\ 7 & 0 & 7 \end{pmatrix} \overset{R_3+R_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ -7 & 6 & -1 \\ 0 & 6 & 6 \end{pmatrix} \overset{R_3/6}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ -7 & 6 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

$\overset{R_2+R_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ -7 & 7 & 0 \\ 0 & 1 & 1 \end{pmatrix} \overset{R_2/7}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ Let $x_2=c$

$v_1 = \begin{pmatrix} c \\ c \\ -c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\underline{\lambda_2 = 2}$

$$\begin{pmatrix} -6 & 0 & 0 \\ -7 & 0 & -1 \\ 7 & 0 & 1 \end{pmatrix} \underset{R_2+R_3}{\overset{R_1/-1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 7 & 0 & 1 \end{pmatrix} \Rightarrow x_1 = 0 \Rightarrow x_3 = 0$$

Let $x_2 = d$

$$\Rightarrow v_2 = \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} = d\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda_3 = 3}$

$$\begin{pmatrix} -7 & 0 & 0 \\ -7 & -1 & -1 \\ 7 & 0 & 0 \end{pmatrix} \underset{\substack{R_2/-1 \\ R_1/-7}}{\overset{R_3+R_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_2-7R_1}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_1 = 0$. Let $x_2 = e \Rightarrow x_3 = -e$

$$\Rightarrow v_3 = \begin{pmatrix} 0 \\ e \\ -e \end{pmatrix} = e\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(18) $T(x,y) = (2x-3y, -3x+2y) = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 9 = \lambda^2 - 4\lambda - 5$$
$$= (\lambda+1)(\lambda-5) = 0$$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = 5$

$\underline{\lambda_1 = -1}$

$$\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \underset{R_1/3}{\overset{R_2+R_1}{\sim}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{Let } x_1 = c$$

$\Rightarrow v_1 = \begin{pmatrix} c \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 5}$

$$\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \underset{R_1/-3}{\overset{R_2-R_1}{\sim}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{Let } x_1 = d$$

$\Rightarrow v_2 = \begin{pmatrix} c \\ -c \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(23) (a) False | (e) False | (g) False

__Ctex:__ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | Any square matrix : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A 3×3 matrix M has e.vals 1, 2, & 3. What is the coeff. of λ^2 is the char. poly. ?

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

$$= (1-\lambda)(6-5\lambda+\lambda^2)$$
$$= 6-5\lambda+\lambda^2-6\lambda+5\lambda^2-\lambda^3$$
$$= -\lambda^3+6\lambda^2-5+6$$

$\boxed{6}$

5.2) 2

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 12-7\lambda+\lambda^2-2 = \lambda^2-7\lambda+10$$
$$= (\lambda-2)(\lambda-5) = 0$$

$\Rightarrow \lambda_1 = 2, \ \lambda_2 = 5$

$\underline{\lambda_1 = 2}$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \overset{R_2-R_1}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \qquad \text{Let } x_2 = c$$

$$\Rightarrow v_1 = \begin{pmatrix} -2c \\ c \end{pmatrix} = c\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = 5}$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \underset{R_2 - R_1}{\overset{R_1/-2}{\sim}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{Let } x_1 = c$$

$$\Rightarrow v_2 = \begin{pmatrix} c \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $C = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\det C = -2 - 1 = -3 \neq 0$

So C is invertible.

Now $C^{-1} = \frac{1}{\det C}\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$

Thus: $D = \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} -6+2 & 3+2 \\ -2+4 & 1+4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 2 & 5 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 4+2 & -5+5 \\ -4+4 & 5+10 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 & 0 \\ 0 & 15 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

(6) $\begin{vmatrix} -3-\lambda & 5 & -20 \\ 2 & -\lambda & 8 \\ 2 & 1 & 7-\lambda \end{vmatrix} = (-3-\lambda)[-\lambda(7-\lambda)-8]$
$-5[2(7-\lambda)-16]+(-20)[2+2\lambda]$

$= (-3-\lambda)(\lambda^2-7\lambda-8)-5(-2\lambda-2)-(40+40\lambda)$

$= -3\lambda^2+21\lambda+24-\lambda^3+7\lambda^2+8\lambda+10\lambda+10-40-40\lambda$

$= -\lambda^3+4\lambda^2-\lambda-6 = (\lambda+1)(\lambda-2)(\lambda-3)=0$

$\lambda_1=-1,\ \lambda_2=2,\ \lambda_3=3$

$\underline{\lambda_1=-1}$

$$\begin{pmatrix} -2 & 5 & -20 \\ 2 & 1 & 8 \\ 2 & 1 & 8 \end{pmatrix} \underset{R_1+R_2}{\overset{R_3-R_2}{\sim}} \begin{pmatrix} 0 & 6 & -12 \\ 2 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1/6}{\sim} \begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R_2-R_1}{\sim} \begin{pmatrix} 0 & 1 & -2 \\ 2 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_2/5}{\sim} \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$ Let $x_3=c$

$$\Rightarrow v_1 = \begin{pmatrix} -5c \\ 2c \\ c \end{pmatrix} = c\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2=2}$

$$\begin{pmatrix} -5 & 5 & -20 \\ 2 & -2 & 8 \\ 2 & 1 & 5 \end{pmatrix} \underset{R_2/2}{\overset{R_1/5}{\sim}} \begin{pmatrix} -1 & 1 & -4 \\ 1 & -1 & 4 \\ 2 & 1 & 5 \end{pmatrix} \overset{R_1+R_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 4 \\ 2 & 1 & 5 \end{pmatrix}$$

$$\overset{R_2+R_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 9 \\ 2 & 1 & 5 \end{pmatrix} \overset{R_2/3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} \overset{R_3-2R_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$\Rightarrow x_1=0$ Let $x_2=d$ $v_2=\begin{pmatrix} 0 \\ d \\ d \end{pmatrix} = d\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_3 = 3}$

$$\begin{pmatrix} -6 & 5 & -20 \\ 2 & -3 & 8 \\ 2 & 1 & 4 \end{pmatrix} \underset{R_1+3R_2}{\overset{R_3-R_2}{\sim}} \begin{pmatrix} 0 & -4 & 4 \\ 2 & -3 & 8 \\ 0 & 4 & -4 \end{pmatrix} \underset{R_3/4}{\overset{R_1+R_3}{\sim}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 8 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\overset{R_2+3R_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{Let } x_3 = 2e$$

$$\Rightarrow v_3 = \begin{pmatrix} -5e \\ 2e \\ 2e \end{pmatrix} = e\begin{pmatrix} -5 \\ 2 \\ 2 \end{pmatrix}$$

Let $C = \begin{pmatrix} -5 & 0 & -5 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

$\det C = -5(2-2) + 0 - 5(2-1) = -5 \neq 0$

Then $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

13 d) False | f) False | i) False

$\underline{\text{Ctex:}}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

[redacted]

Thm 5.1 Let A be an $n \times n$ matrix.
1) If $Av = \lambda v$, then $A^k v = \lambda^k v$.
2) $(\det A \neq 0 \text{ \& } Av = \lambda v) \Rightarrow (\lambda \neq 0 \text{ \& } A^{-1}v = \frac{1}{\lambda} v$.

pf/

1) By Induction

$$Av = \lambda v$$

Base: $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$

Assume: $A^k v = \lambda^k v$

Then

$$A^{k+1} v = A(A^k v) = A(\lambda^k v) = \lambda^k (Av) = \lambda^{k+1} v$$

$\square$

2) $\det A \neq 0 \Rightarrow A^{-1}$ exists

We know $v \neq 0$, by def.

$$Av = \lambda v \Rightarrow A^{-1}(Av) = A^{-1}(\lambda v)$$

$$\Rightarrow (A^{-1}A)v = \lambda(A^{-1}v) \Rightarrow v = \lambda A^{-1} v$$

$$\Rightarrow \lambda \neq 0 \text{ \& } \frac{1}{\lambda} v = A^{-1} v$$

$\square$